STATISTICS FOR ASTRONOMY 2022–2023
RE-EXAMINATION
2 February 2023 (15.00 - 17.00)

DIRECTIONS: **Allow 2 hours.** Write your name and student number at the top of every page of your solutions. Please explain clearly all of the steps that you used to derive a result. Please make certain that your handwriting is readable to someone besides yourself.

1. Answer the following open questions. **(7 points/question)**

   (a) Consider the case that *Jeffrey's prior* is used for prob$(x \mid I)$, the prior probability for variable x. Use the transformation rule to calculate prob $(\log(x) \mid I)$, the probability distribution function of the (natural) logarithm of x, from prob$(x \mid I)$.

   (b) Using Cox's product and/or sum rules, derive the mathematical formula that describes Bayes' theorem. Explain why the denominator in Bayes' formula is often referred to as the "evidence".

   (c) Describe and compare *parameter estimation* and *model comparison*. What role does the evidence play in each process?

   (d) Describe the *principle of indifference* and give a simple example of its application to determine a probability distribution function.

   (e) Given a continuous distribution function prob$(x) = \frac{1}{2}x$ for $0 \leq x \leq 2$ and prob$(x) = 0$ otherwise, calculate the expectation value $\langle x \rangle$ and the variance of x, Var(x).

   (f) Given a one-dimensional probability density function, describe how you would find and compute the 95% confidence interval.

   (g) There are several ways to discover exoplanets, including radial velocity searches. In a radial velocity search, an exoplanet is found by measuring the change of the radial velocity of its host star. Such an exoplanet is often characterized by the period P and eccentricity e of its orbit. These aren't the only parameters that need to be inferred: the model of a planet's orbit *also* includes the mean radial velocity of the star V_0, the velocity amplitude of the orbit K, the "longitude of the periastron" ω, and the fraction of the orbit prior to the start of the data taking at which the periastron occurred χ. Given some data **D** (which includes the velocities of the host star measured at some Julian Dates).

      i. name and describe the process by which you could estimate the *joint probability distribution function* prob$(P, e \mid \mathbf{D}, I)$ from the full posterior probability distribution prob$(V_0, K, P, e, \omega, \chi \mid \mathbf{D}, I)$ determined from the orbital model, and

      ii. describe how you would determine the *best estimates* of the period P_0 and eccentricity e_0 of the orbit and the *uncertainties* on these best estimates σ_P and σ_e.

   (h) Consider the following problem: a dataset of N values, $\{y_i \pm \sigma_i\}$ (with $1 \leq i \leq N$) and associated positions $\{x_i\}$, is fitted to the model $Y(x) = a\sin(x) + b\cos(x)$, thereby solving for a and b. Write down the corresponding χ^2 function for this fitting problem.

→ **See next pages for questions 2 and 3**

2 Open questions involving derivations

(a) **(16 points)** Given a data set $\{x_i \; 0 \leq i < N\}$ of N values, where each value x_i is independent and drawn from the same normal (Gaussian) distribution with known (constant) standard deviation σ, use Bayes' theorem to prove that the most probable value μ_0 for the mean of the set is the average of the data Assume a flat prior for the mean μ

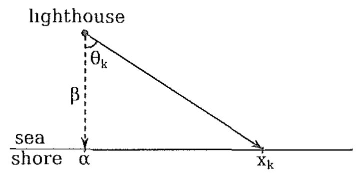

Figure 1 The Lighthouse problem

(b) **(14 points)** Consider "the lighthouse problem" (Fig 1) A lighthouse at sea is at a position α along the coast and a distance β at sea The shore is lined with photodetectors that record only that a flash has been recorded, but not the angle from which it came N flashes have been recorded at positions $\{x_i \; 0 \leq i < N\}$ along the coast, corresponding with azimuth directions $\{\theta_i \; 0 \leq i < N\}$ The relation between a detected position and azimuth direction is given by $\beta \tan\theta_i = x_i - \alpha$ We know that the lighthouse flashes uniformly in the directions of interest $\mathrm{prob}(\theta_i|\alpha,\beta,I) = \frac{1}{\pi}$ for $-\frac{\pi}{2} \leq \theta_i \leq \frac{\pi}{2}$ Write down the transformation rule, and use it to calculate $\mathrm{prob}(x_i|\alpha,\beta,I)$

Hint the table below contains some of the common trigonometric derivatives

function	derivative	function	derivative
$\sin x$	$\cos x$	$\arcsin x$	$1/\sqrt{1-x^2}$
$\cos x$	$-\sin x$	$\arccos x$	$-1/\sqrt{1-x^2}$
$\tan x$	$\sec^2 x$	$\arctan x$	$1/(1+x^2)$

(c) **(4 points)** The probability density function derived in the previous question is described by two different distribution functions What are the names of these distribution functions and how are they related?

→ **See next page for question 3**